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THE DIRICHLET NORM AND THE NORM OF SZEGÖ TYPE[1]

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ABSTRACT. Let S be a smoothly bounded region in the complex plane. Let $g(z, t)$ denote the Green's function of S with pole at t. We show that

$$\iint_S |f'(z)|^2\, dx\, dy < \frac{1}{2}\int_{\partial S} |f'(z)|^2 \left(\frac{\partial g(z, t)}{\partial n_z}\right)^{-1} |dz|$$

holds for any analytic function $f(z)$ on $S \cup \partial S$. This curious inequality is obtained as a special case of a much more general result.

1. Introduction and preliminary facts. Let S denote an arbitrary compact bordered Riemann surface with boundary contours $\{C_\nu\}_{\nu=2g+1}^{2g+m}$ and of genus g. Let $\{C_\nu\}_{\nu=1}^{2g+m-1}$ be a canonical homology basis for S. Let $W(z, t)$ denote a meromorphic function whose real part is the Green's function $g(z, t)$ with pole at $t \in S$. The differential id $W(z, t)$ is positive along ∂S and has $N = 2g + m - 1$ zeros $\{t_\nu\}$ in S. We *assume* that the points t_ν are simple and they are not on $\{C_\nu\}_{\nu=1}^{2g}$; the other cases will require only a slight modification. For simplicity, we do not distinguish between points $z \in S \cup \partial S$ and local parameters z. For an arbitrary integer q and for any positive continuous function $\rho(z)$ on ∂S, we let $H^q_{p,\rho}(S)$ $[p > 1]$ be the Banach space of analytic differentials $f(z)(dz)^q$ of order q on S with a finite norm

$$\left\{\frac{1}{2\pi}\int_{\partial S} |f(z)(dz)^q|^p \rho(z)\left[\text{id } W(z, t)\right]^{1-qp}\right\}^{1/p} < \infty,$$

where $f(z)$ means the Fatou boundary value of f at $z \in \partial S$. Let $K_{q,t,\rho}(z, \bar{u})(dz)^q$ be the reproducing kernel of $H^q_{2,\rho}(S)$ which is characterized by the reproducing property

$$f(u) = \frac{1}{2\pi}\int_{\partial S} f(z)(dz)^q\, \overline{K_{q,t,\rho}(z, \bar{u})(dz)^q}\, \rho(z)\left[\text{id } W(z, t)\right]^{1-2q}$$

for all $f(z)(dz)^q \in H^q_{2,\rho}(S)$ (see [5]). Let $L_{q,t,\rho}(z, u)(dz)^{1-q}$ denote the adjoint L-kernel of $K_{q,t,\rho}(z, \bar{u})(dz)^q$. Then, $L_{q,t,\rho}(z, u)(dz)^{1-q}$ is a meromorphic differential on S of order $1 - q$ with a simple pole at u having residue 1.

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[1]Dedicated to my father on his 75th birthday.

Moreover:

$$\overline{K_{q,t,\rho}(z,\bar{u})(dz)^q}\,\rho(z)[\mathrm{id}\,W(z,t)]^{1-2q} = (1/i)L_{q,t,\rho}(z,u)(dz)^{1-q} \quad \text{along } \partial S. \tag{1.1}$$

We note that $K_{q,t,\rho}(z,\bar{u})$ and $L_{q,t,\rho}(z,u)$ are continuous along ∂S. If S is a bounded regular region *in the plane*, we can define these kernels for arbitrary real values of q (cf. [**5**, §§2 and 6]).

Next, let $K^E(z,\bar{u})$ and $L^E(z,u)$ denote the exact Bergman kernel and its adjoint L-kernel on S, respectively (cf. [**8**, p. 117]). $L^E(z,u)dz$ is analytic on $S \cup \partial S$ except for u, where it has a double pole

$$\left\{\frac{1}{\pi}\frac{1}{(z-u)^2} + \text{regular terms}\right\} dz. \tag{1.2}$$

Furthermore, it satisfies the relation

$$\overline{-K^E(z,\bar{u})dz} = L^E(u,z)\,dz \quad \text{along } \partial S. \tag{1.3}$$

Let $Z_\nu(z) = \int_{C_\nu} L(\zeta, z)\,d\zeta$. Then $\{Z_\nu(z)\,dz\}_{\nu=1}^N$ is a basis for the analytic differentials on S which are real along ∂S. Here $L(\zeta, z)$ is the adjoint L-kernel of the usual Bergman kernel $K(\zeta,\bar{z})$ on S (cf. [**8**, §§4.3, 4.5 and 4.10]). Then from (1.1) and (1.3), we obtain

$$K_{q,t,\rho}(z,\bar{u})K_{1-q,t,\rho^{-1}}(z,\bar{u}) = \pi K^E(z,\bar{u}) + \sum_{\nu=1}^{N}\sum_{\mu=1}^{N} C_{\nu,\mu}\overline{Z_\nu(u)}\,Z_\mu(z) \tag{1.4}$$

and

$$L_{q,t,\rho}(z,u)L_{1-q,t,\rho^{-1}}(z,u) = \pi L^E(u,z) - \sum_{\nu=1}^{N}\sum_{\mu=1}^{N} \overline{C_{\nu,\mu}}\,Z_\nu(u)\,Z_\mu(z), \tag{1.5}$$

where the constants $C_{\nu,\mu}$ are uniquely determined as in [**5**].

2. The main theorem. Let $\{\Phi_j(z)(dz)^q\}_{j=1}^\infty$ and $\{\Psi_j(z)(dz)^{1-q}\}_{j=1}^\infty$ be complete orthonormal systems for $H^q_{2,\rho}(S)$ and $H^{1-q}_{2,\rho^{-1}}(S)$, respectively. Let $H = H^q_{2,\rho}(S) \otimes H^{1-q}_{2,\rho^{-1}}(S)$ denote the direct product of $H^q_{2,\rho}(S)$ and $H^{1-q}_{2,\rho^{-1}}(S)$. The space H is composed of differentials $f(z_1,z_2)(dz_1)^q(dz_2)^{1-q}$ on $S \times S$ such that

$$f(z_1,z_2) = \sum_{j=1}^\infty\sum_{k=1}^\infty A_{j,k}\Phi_j(z_1)\Psi_k(z_2), \qquad \sum_{j=1}^\infty\sum_{k=1}^\infty |A_{j,k}|^2 < \infty. \tag{2.1}$$

The scalar product $(\ ,\ )_H$ is introduced as follows:

$$(f,h)_H = \sum_{j=1}^\infty\sum_{k=1}^\infty A_{j,k}\overline{B_{j,k}},$$

where

$$h(z_1,z_2) = \sum_{j=1}^\infty\sum_{k=1}^\infty B_{j,k}\Phi_j(z_1)\Psi_k(z_2) \quad \text{and} \quad \sum_{j=1}^\infty\sum_{k=1}^\infty |B_{j,k}|^2 < \infty$$

(cf. [**1**, §8]).

THEOREM 2.1. *Suppose that*

$$f(z_1, z_2)(dz_1)^q(dz_2)^{1-q} = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} A_{j,k}\Phi_j(z_1)\Psi_k(z_2)(dz_1)^q(dz_2)^{1-q} \in H.$$

Then $f(z, z)$ can be uniquely decomposed as follows:

$$f(z, z) = h'(z) + \sum_{\nu=1}^{N} d_\nu Z_\nu(z) \quad \text{for } z \in S. \tag{2.2}$$

It is understood that the d_ν are constants, $h(z)$ is analytic on S, and

$$\iint_S |h'(z)|^2\, dx\, dy < \infty \qquad (z = x + iy).$$

In addition,

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |A_{j,k}|^2$$

$$\geqslant \min \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \left\{ \frac{1}{2\pi}\int_{\partial S} \varphi_j(z_1)(dz_1)^q\, \overline{\varphi_k(z_1)(dz_1)^q} \right.$$

$$\cdot \rho(z_1)[\mathrm{id}\, W(z_1, t)]^{1-2q} \cdot \frac{1}{2\pi}\int_{\partial S} \psi_j(z_2)(dz_2)^{1-q}$$

$$\left. \cdot\, \overline{\psi_k(z_2)(dz_2)^{1-q}}\, (\rho(z_2))^{-1}[\mathrm{id}\, W(z_2, t)]^{2q-1} \right\}$$

$$= \frac{1}{\pi}\iint_S |h'(z)|^2\, dx\, dy + \sum_{\nu=1}^{N}\sum_{\mu=1}^{N} D_{\nu,\mu} d_\nu \overline{d_\mu}, \tag{2.3}$$

where $\|D_{\nu,\mu}\|$ is the inverse of $\|C_{\nu,\mu}\|$. The minimum is taken here over all analytic functions $\sum_{j=1}^{\infty} \varphi_j(z_1)\psi_j(z_2)$ on $S \times S$ satisfying

$$f(z, z) = \sum_{j=1}^{\infty} \varphi_j(z)\psi_j(z) \quad \text{on } S, \tag{2.4}$$

$$\varphi_j(z)(dz)^q \in H^q_{2,\rho}(S) \quad \text{and} \quad \psi_j(z)(dz)^{1-q} \in H^{1-q}_{2,\rho^{-1}}(S).$$

PROOF. The crucial ingredient in this proof is the observation that $\|C_{\nu,\mu}\|$ is positive definite (cf. equation (1.4) and [5, p. 549]). Refer to the proof of Theorem 2.1 in [6]. The positive definiteness of $\|C_{\nu,\mu}\|$ implies that

$$k(z, \bar{u}) = \sum_{\nu=1}^{N}\sum_{\mu=1}^{N} C_{\nu,\mu}\, \overline{Z_\nu(u)}\, Z_\mu(z)$$

is a reproducing kernel for the finite dimensional class $\mathfrak{F}_2$ which is generated by $\{Z_\nu(z)\}_{\nu=1}^{N}$ (see [1, pp. 346–347]). The scalar product is given by

$$(f, h)_2 = \sum_{\nu=1}^{N}\sum_{\mu=1}^{N} D_{\nu,\mu}\zeta_\nu \overline{\eta_\mu}$$

for $f(z) = \sum_{\nu=1}^{N} \zeta_\nu Z_\nu(z)$ and $h(z) = \sum_{\nu=1}^{N} \eta_\nu Z_\nu(z)$. Note that $K_{q,t,\rho}(z, \bar{u})K_{1-q,t,\rho^{-1}}(z, \bar{u})dz$ is the reproducing kernel of the space $\mathfrak{F}$ which is formed by restricting the functions in H to the diagonal set $D = \{(z, z)|z \in S\}$ (cf. [**1**, p. 361, Theorem II]). For $f \in \mathfrak{F}$, the norm $\|f\|_{\mathfrak{F}}$ is given by $\min\|h\|_H$ where $h(z_1, z_2)$ ranges over all elements of H whose restriction to D is $f(z)$. Of course, $\|h\|_H$ denotes the norm of h in H.

On the other hand, the space $\mathfrak{F}$ must coincide with the class corresponding kernel function $K_{q,t,\rho}(z, \bar{u})K_{1-q,t,\rho^{-1}}(z, \bar{u})$ when it is considered as *the sum* of the kernel functions $\pi K^E(z, \bar{u})$ and $k(z, \bar{u})$ (see [**1**, pp. 352–357]). We thus obtain the decomposition (2.2). The uniqueness follows from [**8**, pp. 104 and 108].

Finally, from the definition of the norm in H [**1**, pp. 357–361], we have the inequality (2.3).

Using [**1**] and the preceding remark about $\mathfrak{F}$, we immediately obtain

COROLLARY 2.1. *Any analytic function $f(z)$ on S with a finite Dirichlet integral can be represented as a series*

$$f'(z) = \sum_{j=1}^{\infty} \varphi_j(z)\psi_j(z) \quad \text{on } S, \tag{2.5}$$

where $\varphi_j(z)(dz)^q \in H^q_{2,\rho}(S)$ and $\psi_j(z)(dz)^{1-q} \in H^{1-q}_{2,\rho^{-1}}(S)$.

Furthermore, the equation

$$\begin{aligned}\frac{1}{\pi}\iint_S |f'(z)|^2\,dx\,dy \\ = \min \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \Bigg\{ \frac{1}{2\pi}\int_{\partial S} \varphi_j(z_1)(dz_1)^q\, \overline{\varphi_k(z_1)(dz_1)^q} \\ \cdot \rho(z_1)[\mathrm{id}\, W(z_1, t)]^{1-2q} \cdot \frac{1}{2\pi}\int_{\partial S} \psi_j(z_2)(dz_2)^{1-q} \\ \cdot \overline{\psi_k(z_2)(dz_2)^{1-q}}\,(\rho(z_2))^{-1}[\mathrm{id}\, W(z_2, t)]^{2q-1}\Bigg\}\end{aligned} \tag{2.6}$$

is valid. The minimum is taken here over all analytic functions $\sum_{j=1}^{\infty} \varphi_j(z_1)\psi_j(z_2)$ satisfying (2.5).

Conversely, if the jk sum in (2.6) *is finite, then the exact differential $f'(z)dz$ defined by the series* (2.5) *has a finite Dirichlet integral.*

3. Some inequalities. As an application of the main theorem, we derive some inequalities. To start with, consider the case $f(z_1, z_2) = \varphi(z_1)\psi(z_2)$. This leads to

THEOREM 3.1. *For any* $\varphi(z)(dz)^q \in H^q_{2,\rho}(S)$ *and* $\psi(z)(dz)^{1-q} \in H^{1-q}_{2,\rho^{-1}}(S)$, *we have*

$$\frac{1}{2\pi}\int_{\partial S}|\varphi(z_1)(dz_1)^q|^2\rho(z_1)[\mathrm{id}\, W(z_1,t)]^{1-2q}$$
$$\cdot \frac{1}{2\pi}\int_{\partial S}|\psi(z_2)(dz_2)^{1-q}|^2(\rho(z_2))^{-1}[\mathrm{id}\, W(z_2,t)]^{2q-1}$$
$$\geq \min \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left\{\frac{1}{2\pi}\int_{\partial S}\varphi_j(z_1)(dz_1)^q\,\overline{\varphi_k(z_1)(dz_1)^q}\right.$$
$$\cdot\, \rho(z_1)[\mathrm{id}\, W(z_1,t)]^{1-2q}\quad \cdot \frac{1}{2\pi}\int_{\partial S}\psi_j(z_2)(dz_2)^{1-q}$$
$$\left.\cdot\, \overline{\psi_k(z_2)(dz_2)^{1-q}}\,(\rho(z_2))^{-1}[\mathrm{id}\, W(z_2,t)]^{2q-1}\right\}$$
$$= \frac{1}{\pi}\iint_S |h'(z)|^2\, dx\, dy + \sum_{\nu=1}^{N}\sum_{\mu=1}^{N} D_{\nu,\mu} d_\nu \overline{d_\mu}, \tag{3.1}$$

where $\varphi(z)\psi(z) = h'(z) + \sum_{\nu=1}^N d_\nu Z_\nu(z)$ *on* S, *and where the minimum is taken over all analytic functions* $\sum_{j=1}^\infty \varphi_j(z_1)\psi_j(z_2)$ *on* $S \times S$ *such that*

$$\varphi(z)\psi(z) = \sum_{j=1}^{\infty}\varphi_j(z)\psi_j(z). \tag{3.2}$$

Of course, $\varphi_j(z)(dz)^q \in H^q_{2,\rho}(S)$ *and* $\psi_j(z)(dz)^{1-q} \in H^{1-q}_{2,\rho^{-1}}(S)$.

Equality holds in (3.1) *if and only if* $\varphi(z)\psi(z)$ *is expressible in the form* $CK_{q,t,\rho}(z,\bar{u})K_{1-q,t,\rho^{-1}}(z,\bar{u})$ *for some point* $u \in S$ *and for some constant* C.

The equality statement in Theorem 3.1 will be proved in §5.

We can now take $q = 0$, $\rho(z) \equiv 1$, $\varphi(z) \equiv 1$, $\psi(z) \equiv f'(z)$. This yields

COROLLARY 3.1. *For any analytic function* $f(z)$ *on* $S \cup \partial S$, *we have*

$$\iint_S |f'(z)|^2\, dx\, dy \leq \frac{1}{2}\int_{\partial S}\frac{|f'(z)dz|^2}{\mathrm{id}\, W(z,t)}. \tag{3.3}$$

Equality holds in (3.3) *if and only if* S *is simply connected and* $f'(z)$ *is expressible in the form* $C\pi K^E(z,\bar{t}) = CK_{1,t,1}(z,\bar{t})$ *for some constant* C.

Regarding the equality statement in Corollary 3.1, we note that $K_{0,t,1}(z,\bar{u}) \equiv 1$ if and only if $u = t$ [3]. Furthermore, we can compare (3.3) with the inequality

$$\left(\frac{1}{\pi}\iint_S |f'(z)|^2\, dx\, dy\right)^2 = \left(\frac{1}{2\pi i}\int_{\partial S}\overline{f(z)}\, f'(z)\, dz\right)^2$$
$$\leq \frac{1}{2\pi}\int_{\partial S}|f(z)|^2\,\mathrm{id}\, W(z,t)\,\frac{1}{2\pi}\int_{\partial S}\frac{|f'(z)dz|^2}{\mathrm{id}\, W(z,t)}.$$

Let $K^E_{1,t,1}(z, \bar{u})dz$ denote the reproducing kernel of the closed subspace of $H^1_{2,1}(S)$ composed of exact analytic differentials on S (cf. [2] and [4]). Since $L^E(z, u) = L^E(u, z)$ if and only if S is planar [8, pp. 114–120], Theorem 3.3 in [2] requires a modification when $g > 1$. But, this is not difficult. Using Corollary 3.1, we now obtain

COROLLARY 3.2. *For all t and $u \in S$, we have*

$$K^E_{1,t,1}(u, \bar{u}) \leq \pi K^E(u, \bar{u}). \tag{3.4}$$

Equality holds in (3.4) *if and only if S is simply connected and $u = t$.*

PROOF. From (3.3) and the extremal property of $K^E(z, \bar{u})$ [8, pp. 135–137], we have

$$\frac{1}{K^E(u, \bar{u})} = \iint_S \left| \frac{K^E(z, \bar{u})}{K^E(u, \bar{u})} \right|^2 dx\, dy \leq \iint_S \left| \frac{K^E_{1,t,1}(z, \bar{u})}{K^E_{1,t,1}(u, \bar{u})} \right|^2 dx\, dy$$

$$\leq \frac{1}{2} \int_{\partial S} \left| \frac{K^E_{1,t,1}(z, \bar{u})}{K^E_{1,t,1}(u, \bar{u})} \right|^2 \frac{|dz|^2}{\mathrm{id}\, W(z, t)} = \frac{\pi}{K^E_{1,t,1}(u, \bar{u})}. \tag{3.5}$$

We note that (3.3) is, in general, *not valid* for arbitrary analytic differentials $f(z)\, dz$. Indeed, suppose that inequality (3.3) were valid for $K_{1,t,1}(z, \bar{t})\, dz$. Then, from the argument of Corollary 3.2, we would have

$$K_{1,t,1}(t, \bar{t}) \leq \pi K(t, \bar{t}), \tag{3.6}$$

which implies a contradiction for doubly connected regions S (cf. [6, §7]). The relationship between $K^E_{1,t,1}(z, \bar{u})$ and $\pi K^E(z, \bar{u})$ is, in general, quite complicated. (See [4, equation (2.6)].)

On the other hand, by setting $q = 0$ and $\rho(z) \equiv 1$ in Theorem 3.3, we obtain

COROLLARY 3.3. *For the critical points $\{t_\mu\}_{\mu=1}^N$ of the Green's function $g(z, t)$ of G, the matrix*

$$\left\| \sum_{\mu=1}^{N} \frac{Z_\nu(t_\mu) Z_{\nu'}(t_\mu)}{W''(t_\mu, t)} - D_{\nu,\nu'} \right\|^{N \times N}$$

is positive definite. A slight modification is required here when the t_μ are not simple.

PROOF. We consider the decomposition $1 \cdot \sum_{\nu=1}^N d_\nu Z_\nu(z)$ for arbitrary constants d_ν. Then, we have

$$\frac{1}{2\pi} \int_{\partial S} \frac{\left|\sum_{\nu=1}^N d_\nu Z_\nu(z)\, dz\right|^2}{\mathrm{id}\, W(z, t)} > \sum_{\nu=1}^{N} \sum_{\nu'=1}^{N} D_{\nu,\nu'}\, d_\nu\, \overline{d_{\nu'}}. \tag{3.7}$$

Using the residue theorem, we deduce the desired result.

4. Integral transform by $K_{q,t,\rho}(z, \bar{u})K_{1-q,t,\rho^{-1}}(z, \bar{v})$. As another application of the main theorem, we show that all the results of [7] are valid for the present H.

THEOREM 4.1. *Assume that $p > 1$. Then, for $\sigma \in L_p(\partial S)$,*

$$F_\sigma(z_1, z_2) = \int_{\partial S} \sigma(\zeta)\, \overline{K_{q,t,\rho}(\zeta, \bar{z}_1) K_{1-q,t,\rho^{-1}}(\zeta, \bar{z}_2)\, d\zeta} \tag{4.1}$$

belongs to H if and only if the projection $h_1(z)$ of $\sigma(z)$ onto $H^0_{p,1}(S)$ belongs to the Bergman space of S; that is,

$$\iint_S |h_1'(z)|^2\, dx\, dy < \infty. \tag{4.2}$$

PROOF. Refer to the proof of [7, Theorem 2.1]. The necessity is now apparent from Theorem 2.1. The crucial step in the sufficiency is to show that

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left| \int_{\partial S} h_1(z)\, \overline{\Phi_j(z)\Psi_k(z)\, dz} \right|^2 = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left| -2i \iint_S h_1'(z)\, \overline{\Phi_j(z)\Psi_k(z)}\, dx\, dy \right|^2 \tag{4.3}$$

converges. For any double sequence $\{A_{j,k}\}$ satisfying

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |A_{j,k}|^2 < \infty \tag{4.4}$$

we consider the function $f(z_1, z_2)$ defined by (2.1). Then, using Theorem 2.1, we have

$$f(z, z) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} A_{j,k} \Phi_j(z) \Psi_k(z) = \tilde{f}'(z) + \sum_{\nu=1}^{N} \tilde{C}_\nu Z_\nu(z) \quad \text{on } S \tag{4.5}$$

and

$$\iint_S |f(z, z)|^2\, dx\, dy < \infty. \tag{4.6}$$

Note that

$$f_M(z, z) = \sum_{j=1}^{M} \sum_{k=1}^{M} A_{j,k} \Phi_j(z) \Psi_k(z) \tag{4.7}$$

also can be uniquely decomposed as follows:

$$f_M(z, z) = \tilde{f}_M'(z) + \sum_{\nu=1}^{N} \tilde{C}_\nu^{(M)} Z_\nu(z) \quad \text{on } S. \tag{4.8}$$

From the main theorem, we see that the convergence of $f_M(z, z)\, dz$ to $f(z, z)\, dz$ in $\mathfrak{F}$ implies both the convergence of $\tilde{f}_M(z)$ to $\tilde{f}(z)$ in the Dirichlet norm and $\lim_{M\to\infty} \tilde{C}_\nu^{(M)} = \tilde{C}_\nu$, for each ν. We thus obtain

$$\iint_S h_1'(z)\overline{f(z,z)}\,dx\,dy = \iint_S h_1'(z)\overline{\left(\lim_{M\to\infty} f_M(z,z)\right)}\,dx\,dy$$

$$= \iint_S h_1'(z)\overline{\left(\lim_{M\to\infty}\tilde{f}_M'(z) + \lim_{M\to\infty}\sum_{\nu=1}^{N}\tilde{C}_\nu^{(M)}Z_\nu(z)\right)}\,dx\,dy$$

$$= \lim_{M\to\infty}\iint_S h_1'(z)\overline{\left(\tilde{f}_M'(z) + \sum_{\nu=1}^{N}\tilde{C}_\nu^{(M)}Z_\nu(z)\right)}\,dx\,dy$$

$$= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(\iint_S h_1'(z)\overline{\Phi_j(z)\Psi_k(z)}\,dx\,dy\right)\overline{A_{j,k}} \tag{4.9}$$

for any $\{A_{j,k}\}$ satisfying (4.4). Hence, from the Landau theorem, we obtain the desired result.

The proof of this theorem shows that all the results of [7] are, in general, valid for the present H.

Let $H_{D(0)}$ denote the subspace of H formed by those functions in H which vanish along the diagonal set D. Let $(H_{D(0)})^\perp$ denote the orthocomplement of $H_{D(0)}$ in H. As in [7], we obtain

THEOREM 4.2. *Any $f(z_1, z_2)(dz_1)^q(dz_2)^{1-q} \in (H_{D(0)})^\perp$ is expressible in the form*

$$f(z_1, z_2) = \frac{1}{2\pi}\int_{\partial S}\frac{h(\zeta)d\zeta\,\overline{K_{q,t,\rho}(\zeta,\bar{z}_1)K_{1-q,t,\rho^{-1}}(\zeta,\bar{z}_2)\,d\zeta}}{\mathrm{i}d\,W(\zeta,t)} \tag{4.10}$$

for a uniquely determined $h(z)\,dz$ in $H_{2,1}^1(S)$.

Furthermore:

$$h(z) = -W'(z,t)\left\{\int_t^z\left(\sum_{\nu=1}^{N}X_\nu(f)L_{q,t,\rho}(\zeta,t_\nu)L_{1-q,t,\rho^{-1}}(\zeta,t_\nu) + f(\zeta,\zeta)\right)d\zeta\right\}, \tag{4.11}$$

where the constants $X_\nu(f)$ are uniquely determined using the equations

$$\sum_{\nu=1}^{N}X_\nu(f)\int_{C_\mu}L_{q,t,\rho}(\zeta,t_\nu)L_{1-q,t,\rho^{-1}}(\zeta,t_\nu)\,d\zeta$$

$$= -\int_{C_\mu}f(\zeta,\zeta)\,d\zeta, \qquad \mu = 1, 2, \ldots, N. \tag{4.12}$$

5. Proof of the equality statement in Theorem 3.1. As an application of Theorem 4.2, we now prove the equality statement in Theorem 3.1.

If equality holds in (3.1) for $\varphi(z_1)(dz_1)^q\psi(z_2)(dz_2)^{1-q} \in H$, then we have $\varphi(z_1)(dz_1)^q\psi(z_2)(dz_2)^{1-q} \in (H_{D(0)})^\perp$ (cf. [6, equation (3.2)]). Hence, by Theorem 4.2, we have

$$\varphi(z_1)\psi(z_2) = \frac{1}{2\pi}\int_{\partial S}\frac{h(\zeta)d\zeta\,\overline{K_{q,t,\rho}(\zeta,\bar{z}_1)K_{1-q,t,\rho^{-1}}(\zeta,\bar{z}_2)\,d\zeta}}{\mathrm{i}d\,W(\zeta,t)} \tag{5.1}$$

with $h(\zeta)\,d\zeta \in H^1_{2,1}(S)$. Consider any $\varphi^{(0)}(z)(dz)^q$ such that: (i) $\varphi^{(0)}(z) \not\equiv 0$; (ii) $\varphi^{(0)}(z)$ is analytic on $S \cup \partial S$; (iii) $\varphi^{(0)}(z)(dz)^q$ is orthogonal to $\varphi(z)(dz)^q$ in $H^q_{2,\rho}(S)$. From (5.1), we obtain

$$\int_{\partial S} \frac{h(\zeta)d\zeta\, \overline{\varphi^{(0)}(\zeta)K_{1-q,t,\rho^{-1}}(\zeta, \bar z_2)\, d\zeta}}{\mathrm{id}\, W(\zeta, t)} = 0 \quad \text{for all } z_2 \in S$$

and so

$$\int_{\partial S} \frac{h(\zeta)d\zeta\, \overline{\varphi^{(0)}(\zeta)f(\zeta)d\zeta}}{\mathrm{id}\, W(\zeta, t)} = 0, \tag{5.2}$$

for all $f(\zeta)(d\zeta)^{1-q} \in H^{1-q}_{2,\rho^{-1}}(S)$. Hence, from the theorem of Cauchy-Read, we obtain

$$\frac{\overline{h(\zeta)d\zeta}\, \varphi^{(0)}(\zeta)(d\zeta)^q}{\mathrm{id}\, W(\zeta, t)} = g(\zeta)(d\zeta)^q \quad \text{a.e. along } \partial S, \tag{5.3}$$

with $g(\zeta)(d\zeta)^q \in H^q_{2,\rho}(S)$ (cf. [5, p. 549] and [6, equation (3.9)]). From (5.1), we deduce that

$$\varphi(z_1)\psi(z_2) = \overline{\frac{1}{2\pi}\int_{\partial S} \frac{g(\zeta)(d\zeta)^q K_{q,t,\rho}(\zeta, \bar z_1)K_{1-q,t,\rho^{-1}}(\zeta, \bar z_2)\, d\zeta}{\varphi^{(0)}(\zeta)(d\zeta)^q}}\,. \tag{5.4}$$

Using (5.4) and the residue theorem, we obtain

$$\varphi(z_1)\psi(z_2) = \sum_{j=1}^{a}\sum_{k=0}^{b} X_{j,k} \frac{\partial^k (K_{q,t,\rho}(z_1, \bar u_j)K_{1-q,t,\rho^{-1}}(z_2, \bar u_j))}{\partial \bar u_j^k} \tag{5.5}$$

and so

$$\varphi(z_1) = \sum_{j=1}^{a}\sum_{k=0}^{b} Y^{(1)}_{j,k} \frac{\partial^k K_{q,t,\rho}(z_1, \bar u_j)}{\partial \bar u_j^k}, \tag{5.6}$$

$$\psi(z_2) = \sum_{j=1}^{a}\sum_{k=0}^{b} Y^{(2)}_{j,k} \frac{\partial^k K_{1-q,t,\rho^{-1}}(z_2, \bar u_j)}{\partial \bar u_j^k} \tag{5.7}$$

for some points $u_j \in S$ and some constants $\{X_{j,k}\}$, $\{Y^{(1)}_{j,k}\}$ and $\{Y^{(2)}_{j,k}\}$. From (1.1), we now obtain

$$\sum_{j=1}^{a}\sum_{k=0}^{b} \overline{X_{j,k}} \frac{\partial^k (L_{q,t,\rho}(z_1, u_j)L_{1-q,t,\rho^{-1}}(z_2, u_j))}{\partial u_j^k}$$
$$= \sum_{j'=1}^{a}\sum_{k'=0}^{b}\sum_{j''=1}^{a}\sum_{k''=0}^{b} \overline{Y^{(1)}_{j',k'}}\;\overline{Y^{(2)}_{j'',k''}} \frac{\partial^{k'} L_{q,t,\rho}(z_1, u_{j'})}{\partial u_{j'}^{k'}} \frac{\partial^{k''} L_{1-q,t,\rho^{-1}}(z_2, u_{j''})}{\partial u_{j''}^{k''}} \tag{5.8}$$

for all z_1 and $z_2 \in S$. By setting $z_1 = z_2 = z$ and comparing the orders of the poles at each u_j, we see that $X_{j,k} = 0$ for all j and k such that $k \neq 0$, and so

$Y_{j,k}^{(1)} = Y_{j,k}^{(2)} = 0$ for all j and k such that $k \neq 0$. Thus

$$\sum_{j=1}^{a} \overline{X_{j,0}}\, L_{q,t,\rho}(z_1, u_j) L_{1-q,t,\rho^{-1}}(z_2, u_j)$$

$$= \left(\sum_{j'=1}^{a} \overline{Y_{j',0}^{(1)}}\, L_{q,t,\rho}(z_1, u_{j'}) \right) \left(\sum_{j''=1}^{a} \overline{Y_{j'',0}^{(2)}}\, L_{1-q,t,\rho^{-1}}(z_2, u_{j''}) \right) \tag{5.9}$$

for all z_1 and $z_2 \in S$. Without loss of generality, some $X_{j_0,0}$ is *nonzero*. By considering (5.9) as the identity with respect to z_1,

$$\overline{X_{j_0,0}}\, L_{1-q,t,\rho^{-1}}(z_2, u_{j_0}) = \overline{Y_{j_0,0}^{(1)}} \left(\sum_{j''=1}^{a} \overline{Y_{j'',0}^{(2)}}\, L_{1-q,t,\rho^{-1}}(z_2, u_{j''}) \right) \tag{5.10}$$

for all $z_2 \in S$. Therefore $Y_{j'',0}^{(2)} = 0$ for all j'' except j_0 and so $Y_{j',0}^{(1)} = 0$ for all j' except j_0. Hence $X_{j,0} = 0$ for all j except j_0. This yields the desired result:

$$\varphi(z_1)\psi(z_2) = X_{j_0,0} K_{q,t,\rho}(z_1, \bar{u}_{j_0}) K_{1-q,t,\rho^{-1}}(z_2, \bar{u}_{j_0}). \tag{5.11}$$

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